JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **27**, No. 4, November 2014
http://dx.doi.org/10.14403/jcms.2014.27.4.571

VECTOR GENERATORS OF THE REAL CLIFFORD ALGEBRA $C\ell_{0,n}$

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Abstract. In this paper, we present new vector generators of a matrix subalgebra $L_{0,n}$, which is isomorphic to the Clifford algebra $C\ell_{0,n}$, and we obtain the matrix form of inverse of a vector in $L_{0,n}$. Moreover, we consider the solution of a linear equation $xg_2 = g_2x$, where g_2 is a vector generator of $L_{0,n}$.

1. Introduction

Let $\mathbb{R}^{p,q}$ be the standard n-dimensional pseudo-Euclidean space endowed with the quadratic form $Q(v) = \sum_{i=1}^{p} v_i^2 - \sum_{i=p+1}^{p+q} v_i^2$ of signature (p, q) with $p + q = n$. Also, let $C\ell_{p,q}$ be the corresponding real Clifford algebra of $\mathbb{R}^{p,q}$.

The Clifford algebras are isomorphic to some matrix algebras. In particular, we constructed the subalgebra $L_{0,n}(\mathbb{R})$ of the $2^n \times 2^n$ real matrix algebra $M_{2^n}(\mathbb{R})$ for every $n \in \mathbb{N}$ which is isomorphic to the real Clifford algebra $C\ell_{0,n}$ and called it the "OE-construction" [2]. Also, we showed $g_2, g_3, g_7, \ldots, g_{2^n-1}$ are vector generators of $L_{0,n}(\mathbb{R})$ and proved some interesting properties.

In section 2, we will show that $g_2, g_4, g_8, \ldots, g_{2^n}$ are another vector generators of $L_{0,n}(\mathbb{R})$.

In section 3, we will prove some interesting properties of the vector generators $g_2, g_4, g_8, \ldots, g_{2^n}$ comparing with those of vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$. More concretely, we will calculate the determinant

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Received July 09, 2014; Accepted October 06, 2014.
2010 Mathematics Subject Classification: Primary 15A66.
Key words and phrases: Clifford algebra, Clifford groups, matrix representation.
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The work reported in this paper was conducted during the sabbatical year of Kwangwoon University in 2013, and this work was supported by the Gachon University research fund of 2014.

of a linear combination of generators in $L_{0,n}(\mathbb{R})$ which are different from those in [2] and we obtain the matrix form of inverse of a vector in $L_{0,n}$.

In section 4, we will consider the existence of solutions for some simple linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$. In fact, by using the construction of matrix representation in [2], the solution set can be obtained easily in some sense. Furthermore the solution set of the equation can be considered in the Clifford algebra $C\ell_{0,n}$, since $L_{0,n}(\mathbb{R})$ is isomorphic to the Clifford algebra $C\ell_{0,n}$.

2. Generators of the algebra $L_{0,n}(\mathbb{R})$

In [2], we constructed vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$ of the subalgebra $L_{0,n}(\mathbb{R})$ of the $2^n \times 2^n$ real matrix algebra $M_{2^n}(\mathbb{R})$ for every $n \in \mathbb{N}$ and proved some interesting properties. In this section, we will show $g_2, g_4, g_8, \ldots, g_{2^n}$ are another vector generators of $L_{0,n}(\mathbb{R})$ and prove some interesting properties comparing with those of vector generators $g_2, g_3, g_7, \ldots, g_{2^n-1}$. First of all, recall some notations given in [2].

Notation. Let

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, let $K_1 = J$ and

$$K_{m-2} = \begin{pmatrix} O_2 & \cdots & O_2 & J \\ O_2 & \cdots & J & O_2 \\ \vdots & \ddots & \vdots & \vdots \\ J & \cdots & O_2 & O_2 \end{pmatrix} \in M_{2^{m-2}}(\mathbb{R}) \ ,$$

for $4 \leq m \leq n$. Also, let

$$T_{m-1} = \begin{pmatrix} O_{2^{m-2}} & -K_{m-2} \\ K_{m-2} & O_{2^{m-2}} \end{pmatrix} \in M_{2^{m-1}}(\mathbb{R}) \ ,$$

for $3 \leq m \leq n$.

Remark 2.1. By using the above notations, $g_{2^i} \in L_{0,n}(\mathbb{R})$ for $i = 1, 2, \ldots, n$ can be written as follows:

$$g_2 = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix} \in M_{2^n}(\mathbb{R}),$$

$$g_{2^{m-1}} = \begin{pmatrix} T_{m-1} & O_{2^{m-1}} & \cdots & O_{2^{m-1}} \\ O_{2^{m-1}} & T_{m-1} & \cdots & O_{2^{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{m-1}} & O_{2^{m-1}} & \cdots & T_{m-1} \end{pmatrix} \in M_{2^n}(\mathbb{R}),$$

for $3 \leq m \leq n$, and

$$g_{2^n} = \begin{pmatrix} O_2 & \cdots & O_2 & O_2 & \cdots & -J \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O_2 & \cdots & O_2 & -J & \cdots & O_2 \\ O_2 & \cdots & J & O_2 & \cdots & O_2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ J & \cdots & O_2 & O_2 & \cdots & O_2 \end{pmatrix} \in M_{2^n}(\mathbb{R}) .$$

Let $\Gamma = \{g_2, g_{2^2}, \ldots, g_{2^n}\}$. Then, $g_{2^i} \in \Gamma$ has the following properties:

LEMMA 2.2. *Let $g_{2^i} \in \Gamma$. Then,*

(1) *g_{2^i} is antisymmetric for all $i = 1, 2, \ldots, n$.*
(2) *$g_{2^i}^2 = -I_{2^n}$ for all $i = 1, 2, \ldots, n$.*

Proof. (1) It is obvious from the definition of g_{2^i}.
(2) Since $E^2 = -I_2$ and $J^2 = I_2$, we obtain $g_{2^i}^2 = -I_{2^n}$ by straightforward computations. □

Moreover, any two elements in Γ are anticommutative as follows:

PROPOSITION 2.3. *For all $i \geq 2$, we have $g_2\, g_{2^i} = -g_{2^i}\, g_2$.*

Proof. For $i = 2$, $g_2\, g_{2^i} = -g_{2^i}\, g_2$, since $EJ = -JE$.
Now, assume that $g_2\, g_{2^k} = -g_{2^k}\, g_2$. Note that

$$g_2 = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix}, \quad g_{2^k} = \begin{pmatrix} T_k & O_{2^k} & \cdots & O_{2^k} \\ O_{2^k} & T_k & \cdots & O_{2^k} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^k} & O_{2^k} & \cdots & T_k \end{pmatrix}.$$

Set

$$g_2^{(k)} = \begin{pmatrix} E & O_2 & \cdots & O_2 \\ O_2 & E & \cdots & O_2 \\ \vdots & \vdots & \ddots & \vdots \\ O_2 & O_2 & \cdots & E \end{pmatrix} \in M_{2^k}(\mathbb{R}).$$

Then, from the equation $g_2\, g_{2^k} = -g_{2^k}\, g_2$, we have

$$g_2^{(k)} T_k = -T_k g_2^{(k)}.$$

Thus,

$$\begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix} \begin{pmatrix} O_{2^{k-1}} & -K_{k-1} \\ K_{k-1} & O_{2^{k-1}} \end{pmatrix} = - \begin{pmatrix} O_{2^{k-1}} & -K_{k-1} \\ K_{k-1} & O_{2^{k-1}} \end{pmatrix} \begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix},$$

which implies that $g_2^{(k-1)} K_{k-1} = -K_{k-1} g_2^{(k-1)}$.

To prove $g_2\, g_{2^{k+1}} = -g_{2^{k+1}}\, g_2$, it is enough to show that $g_2^{(k+1)} T_{k+1} = -T_{k+1} g_2^{(k+1)}$.

Since

$$g_2^{(k+1)} = \begin{pmatrix} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} \\ O_{2^{k-1}} & O_{2^{k-1}} & g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & g_2^{(k-1)} \end{pmatrix}$$

and

$$T_{k+1} = \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & -K_{k-1} \\ O_{2^{k-1}} & O_{2^{k-1}} & -K_{k-1} & O_{2^{k-1}} \\ O_{2^{k-1}} & K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} \\ K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix},$$

we have

$$\begin{aligned} g_2^{(k+1)} T_{k+1} &= \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & -g_2^{(k-1)} K_{k-1} \\ O_{2^{k-1}} & O_{2^{k-1}} & -g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} \\ O_{2^{k-1}} & g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} \\ g_2^{(k-1)} K_{k-1} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix} \\ &= \begin{pmatrix} O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} & K_{k-1} g_2^{(k-1)} \\ O_{2^{k-1}} & O_{2^{k-1}} & K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} \\ O_{2^{k-1}} & -K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} \\ -K_{k-1} g_2^{(k-1)} & O_{2^{k-1}} & O_{2^{k-1}} & O_{2^{k-1}} \end{pmatrix} \\ &= -T_{k+1} g_2^{(k+1)}. \end{aligned}$$

Thus, the equation $g_2^{(k+1)} T_k = -T_k g_2^{(k+1)}$ holds and so the proposition is proved. □

PROPOSITION 2.4. *For every $i, j \geq 2$ with $i \neq j$, we have $g_{2^i}\, g_{2^j} = -g_{2^j}\, g_{2^i}$.*

Proof. Note that

$$g_{2^i} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix}, \quad g_{2^{i+1}} = \begin{pmatrix} T_{i+1} & O_{2^{i+1}} & \cdots & O_{2^{i+1}} \\ O_{2^{i+1}} & T_{i+1} & \cdots & O_{2^{i+1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+1}} & O_{2^{i+1}} & \cdots & T_{i+1} \end{pmatrix}.$$

To prove the equality $g_{2^i}\, g_{2^{i+1}} = -g_{2^{i+1}}\, g_{2^i}$, it is enough to show that the following identity is satisfied:

$$\begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} T_{i+1} = -T_{i+1} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix}.$$

Since

$$\begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} = \begin{pmatrix} O_{2^{i-1}} & -K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} \\ K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \\ O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} \\ O_{2^{i-1}} & O_{2^{i-1}} & K_{i-1} & O_{2^{i-1}} \end{pmatrix}$$

and

$$T_{i+1} = \begin{pmatrix} O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} \\ O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1} & O_{2^{i-1}} \\ O_{2^{i-1}} & K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} \\ K_{i-1} & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \end{pmatrix},$$

the following equalities hold:

$$\begin{aligned} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix} T_{i+1} &= \begin{pmatrix} O_{2^{i-1}} & O_{2^{i-1}} & K_{i-1}^2 & O_{2^{i-1}} \\ O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} & -K_{i-1}^2 \\ -K_{i-1}^2 & O_{2^{i-1}} & O_{2^{i-1}} & O_{2^{i-1}} \\ O_{2^{i-1}} & K_{i-1}^2 & O_{2^{i-1}} & O_{2^{i-1}} \end{pmatrix} \\ &= -T_{i+1} \begin{pmatrix} T_i & O_{2^i} \\ O_{2^i} & T_i \end{pmatrix}. \end{aligned}$$

Thus, the equality $g_{2^i}\, g_{2^{i+1}} = -g_{2^{i+1}}\, g_{2^i}$ is proved.

Now, we assume that the equality $g_{2^i}\, g_{2^{i+k}} = -g_{2^{i+k}}\, g_{2^i}$ is true for a natural number k. Then we show the equality $g_{2^i}\, g_{2^{i+k+1}} = -g_{2^{i+k+1}}\, g_{2^i}$.

Note that

$$g_{2^i} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix}, \quad g_{2^{i+k}} = \begin{pmatrix} T_{i+k} & O_{2^{i+k}} & \cdots & O_{2^{i+k}} \\ O_{2^{i+k}} & T_{i+k} & \cdots & O_{2^{i+k}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+k}} & O_{2^{i+k}} & \cdots & T_{i+k} \end{pmatrix}.$$

Now, set

$$g_{2^i}^{(i+k)} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix} \in M_{2^{i+k}}(\mathbb{R}).$$

Then, the equality $g_{2^i}\, g_{2^{i+k}} = -g_{2^{i+k}}\, g_{2^i}$ implies that

$$g_{2^i}^{(i+k)}\, T_{i+k} = -T_{i+k}\, g_{2^i}^{(i+k)},$$

and so

$$g_{2^i}^{(i+k-1)} K_{i+k-1} = -K_{i+k-1} g_{2^i}^{(i+k-1)},$$

since

$$T_{i+k} = \begin{pmatrix} O_{2^{i+k-1}} & -K_{i+k-1} \\ K_{i+k-1} & O_{2^{i+k-1}} \end{pmatrix}.$$

Note that

$$g_{2^i}\, g_{2^{i+k+1}} = \begin{pmatrix} T_i & O_{2^i} & \cdots & O_{2^i} \\ O_{2^i} & T_i & \cdots & O_{2^i} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^i} & O_{2^i} & \cdots & T_i \end{pmatrix} \begin{pmatrix} T_{i+k+1} & O_{2^{i+k+1}} & \cdots & O_{2^{i+k+1}} \\ O_{2^{i+k+1}} & T_{i+k+1} & \cdots & O_{2^{i+k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ O_{2^{i+k+1}} & O_{2^{i+k+1}} & \cdots & T_{i+k+1} \end{pmatrix}$$

and

$$T_{i+k+1} = \begin{pmatrix} O_{2^{i+k-1}} & O_{2^{i+k-1}} & O_{2^{i+k-1}} & -K_{i+k-1} \\ O_{2^{i+k-1}} & O_{2^{i+k-1}} & -K_{i+k-1} & O_{2^{i+k-1}} \\ O_{2^{i+k-1}} & K_{i+k-1} & O_{2^{i+k-1}} & O_{2^{i+k-1}} \\ K_{i+k-1} & O_{2^{i+k-1}} & O_{2^{i+k-1}} & O_{2^{i+k-1}} \end{pmatrix}.$$

Thus, the equality

$$g_{2^i}\, g_{2^{i+k+1}} = -g_{2^{i+k+1}}\, g_{2^i}$$

holds, since $g_{2^i}^{(i+k-1)} K_{i+k-1} = -K_{i+k-1} g_{2^i}^{(i+k-1)}$. □

From Lemma 2.2 (2), Proposition 2.3 and Proposition 2.4, we obtain the main result as follows:

THEOREM 2.5. *$g_2, g_4, g_8, \ldots, g_{2^n}$ are vector generators of $L_{0,n}(\mathbb{R})$, which is isomorphic to the Clifford algebra $C\ell_{0,n}$.*

REMARK 2.6. Recall the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$ defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let $\sigma_4 = \sigma_1\sigma_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. From the process of "OE-construction" given in [2], we can express g_2, g_4, g_8 in $L_{0,3}(\mathbb{R})$ by means of tensor products. That is to say,

$$g_2 = I_2 \otimes I_2 \otimes \sigma_4 \ , \quad g_4 = I_2 \otimes \sigma_4 \otimes \sigma_1 \ , \quad g_8 = \sigma_4 \otimes \sigma_1 \otimes \sigma_1.$$

Since $\sigma_1^2 = I_2, \sigma_4^2 = -I_2$ and $\sigma_1\sigma_4 = -\sigma_4\sigma_1$, we obtain that $g_{2^i}^2 = -I_8$, and $g_{2^i}g_{2^j} = -g_{2^j}g_{2^i}$ for $i \neq j$.

3. Inverse of vectors in $L_{0,n}(\mathbb{R})$

In this section, we will present the matrix representation of the inverse of a vector in $L_{0,n}(\mathbb{R})$. Set $\Delta = \{\sum_{i=1}^n a_i g_{2^i} : a_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$, the set of all vectors of $L_{0,n}(\mathbb{R})$ in section 2. Then, the matrix $A \in \Delta$ satisfies some interesting properties as follows:

Proposition 3.1. *Let $A = \sum_{i=1}^n a_i g_{2^i} \neq O_{2^n} \in \Delta$. Then,*

(1) $\det(A) = \pm(\sum_{i=1}^n a_i^2)^{2^{n-1}}$.

(2) $A^{-1} = \frac{-1}{\sum_{i=i}^n a_i^2} A$.

Proof. (1) Since g_{2^i} is antisymmetric, $A^T = -\sum_{i=1}^n a_i g_{2^i}$. Also, by proposition 2.3, 2.4,

$$AA^T = (\sum_{i=1}^n a_i^2) I_{2^n}.$$

Thus,

$$\det(A)^2 = (\sum_{i=1}^n a_i^2)^{2^n}$$

and so

$$\det(A) = \pm(\sum_{i=1}^n a_i^2)^{2^{n-1}}.$$

(2) Since $AA^T = (\sum_{i=1}^n a_i^2) I_{2^n}$ and $A^T = -A$, the identity $A^{-1} = \frac{-1}{\sum_{i=1}^n a_i^2} A$ can be obtained. □

Example 3.2. *Let $n = 7$ and $A = g_2 - 3g_{16} + 2g_{64} + g_{128} \in L_{0,7}(\mathbb{R})$. Then*

$$\det(A) = (1^2 + (-3)^2 + 2^2 + 1^2)^{2^6} = 15^{64}.$$

Note that by using proposition 3.1, we can show that $A = \sum_{i=1}^n a_i g_{2^i} \neq O_{2^n} \in \Delta$ is an element in the Clifford group.

4. Existence of solutions for a linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$.

Now, we will consider the existence of solutions for a simple linear equation $xa = ax$ in $L_{0,n}(\mathbb{R})$. In fact, by using the matrix representation in [2], the solution set can be obtained easily in some sense. Furthermore, the solution set of the equation can be considered in the Clifford algebra $C\ell_{0,n}$, since $L_{0,n}(\mathbb{R})$ is isomorphic to the Clifford algebra $C\ell_{0,n}$.

Theorem 4.1. *For $g_2 \in \Delta$, the equation $xg_2 = g_2x$ has solutions in $L_{0,n}(\mathbb{R})$ and the solution set of the equation in $L_{0,n}(\mathbb{R})$ is*

$$\left\{ \sum_{m=0}^{2^{n-2}-1} a_m g_{4m+1} + \sum_{m=0}^{2^{n-2}-1} b_m g_{4m+2} \mid a_m, b_m \in \mathbb{R} \right\}.$$

Proof. Let

$$x = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \in M_{2^n}(\mathbb{R}) \ ,$$

where $x_{ij} \in M_2(\mathbb{R})$ for all $1 \leq i, j \leq n$.

Then, the equation $xg_2 = g_2x$ is equivalent with $x_{ij}E = Ex_{ij}$ for all $1 \leq i, j \leq n$. Then we have

$$x_{ij} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

for some $a, b \in \mathbb{R}$, which implies that all the entries x_{i1} of the first column of x are of odd type. From the construction of $L_{0,n}$ in [2], we obtain $x_{2j1} = O_2$ for all $j = 1, 2, \ldots, n$. Thus, x is expressed by

$$x = \sum_{m=0}^{2^{n-2}-1} a_m g_{4m+1} + \sum_{m=0}^{2^{n-2}-1} b_m g_{4m+2},$$

for some $a_m, b_m \in \mathbb{R}$. □

Example 4.2. *Let $n = 3$. Then, the equation $xg_2 = g_2x$ has solutions in $L_{0,3}(\mathbb{R})$ and the solution set of the equation in $L_{0,3}(\mathbb{R})$ is*

$$\{a_0g_1 + b_0g_2 + a_5g_5 + b_6g_6 \mid a_0, a_1, b_0, b_1 \in \mathbb{R}\}.$$

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